# Skin effect in rectangular conductors



1976 J. Phys. A: Math. Gen. 9 669

(http://iopscience.iop.org/0305-4470/9/4/023)

J. Phys. A: Math. Gen., Vol. 9, No. 4, 1976. Printed in Great Britain. 

# Skin effect in rectangular conductors

Kazimierz Jakubiuk and Paweł Zimny

Department of Electrical Engineering, Technical University, 80-952 Gdańsk, ul. Majakowskiego 11/12, Poland

Received 25 July 1975, in final form 24 October 1975

**Abstract.** By means of the conformal representation method the distribution of surface current density is found in a conductor treated as a superconductor. The current density distribution is determined in a rectangular conductor for an arbitrary current shape. The possibility of accepting a one-dimensional model has been demonstrated.

## Notation

| Symbol | Unit | Physical parameter |
|---|---|---|
| $\boldsymbol{A}$ | $\mathrm{V\,s\,m^{-1}}$ | vectorial magnetic potential |
| $\boldsymbol{B}$ | $\mathrm{V\,s\,m^{-2}}$ | magnetic flux density |
| $D, a, k$ | — | conductors shape coefficients |
| $b, h$ | m | conductor cross-sectional dimensions |
| $C_{mn}$ | — | coefficients of a series |
| $F(k), E(k)$ | — | complete elliptic integrals of the first and second kind respectively |
| $F(k, v), E(k, v)$ | — | elliptic integrals of the first and second kind respectively |
| $d$ | $\mathrm{A\,s^{-1}}$ | steepness of current rise |
| $I, I_m$ | A | current |
| $i$ | — | dimensionless current density |
| $J$ | $\mathrm{A\,m^{-2}}$ | reference current density |
| $j$ | $\mathrm{A\,m^{-2}}$ | current density |
| $j_p$ | $\mathrm{A\,m^{-1}}$ | surface current density |
| $L$ | — | area edge |
| $\boldsymbol{n}$ | — | normal vector |
| $ds$ | m | differential of length |
| $t$ | s | time |
| $X, Y, Z$ | m | Cartesian coordinates |
| $x, y, z$ | — | dimensionless coordinates |
| $u, v, w$ | — | complex variables |
| $\alpha$ | rad | angle |
| $\beta$ | — | coefficient |
| $\delta(x)$ | — | Dirac's function |
| $\sigma$ | $\Omega^{-1}\mathrm{m}^{-1}$ | electric conductivity of the conductor |
| $\mu_0$ | $\mathrm{V\,s\,A^{-1}m^{-1}}$ | magnetic permeability of free space |
| $\Phi$ | $\mathrm{V\,s\,m^{-1}}$ | complex magnetic potential |
| $\tau$ | — | dimensionless time |
| $\omega$ | $\mathrm{rad\,s^{-1}}$ | pulsation |

## 1. Introduction

Many authors have investigated the current density distribution in a rectangular conductor (Dwight 1918, Forbes and Gorman 1933, Kennelly *et al* 1915, Strutt 1927). However, they have not given a complete solution of the problem. The increased use of rectangular conductors, especially in such impulse systems as explosion of conductors, makes it necessary to determine the distribution of the current density in order to study the physical phenomena that occur in such cases. If the dimensions of the conductor cross section fulfil the following condition:

$$2b/2h \gg 1 \tag{1}$$

we will call the conductor a foil. The current density distribution in a foil has been determined by Zimny (1970). If the dimensions of the conductor do not fulfil the condition (1) or if the duration of the current impulse is comparable with the expression $\mu_0\sigma h^2$, then a two-dimensional conductor model is taken into consideration in determining the current density distribution.

In this paper the current density distribution in a rectangular conductor is determined for a rectangular surge. Further, using Duhamel's theorem, a solution is found for an arbitrary forcing current and in particular for a linear rising surge and for harmonic current. This is done in two stages.

In the first stage the current density distribution is determined for a rectangular surge. Then using this solution and Duhamel's theorem, in the second stage the current density distribution is determined for a current surge of any shape. This procedure eliminates the basic difficulty, namely, that of solving the skin effect of the surge and simultaneously determining the magnetic field in the inner and outer regions of the conductor.

In the case of a rectangular surge of current flow through the conductor it is possible to ascertain that the magnetic field does not penetrate into the conductor region at the initial moment, to be called $0_+$ henceforth. This is justified by the fact that a rectangular shock may be considered as a sinusoidal excitation with a frequency approaching infinity, and it is known that for very high frequencies the conductor behaves like a conductor placed in an electrostatic field (Kelvin 1890) or as a superconductor (Landau and Lifshitz 1960). We take advantage of this and first find the magnetic field distribution around the superconductor, in which direct current flows. If we know that distribution we can then determine the surface current density and formulate the initial conditions for a rectangular shaped surge. Then, using Maxwell's equations we find the time distribution of the current density for a rectangular surge.

## 2. Current density distribution in superconductors

It is a characteristic feature of a current-carrying superconductor that magnetic field nulling occurs in the superconducting region (Landau and Lifshitz 1960) which makes the current flow in a surface layer of thickness about $10^{-7}$ m. In order to simplify the problem we assume that the current flows on the surface and we introduce the term surface current density.

The vectorial magnetic potential on the outer side of a long rectangular superconductor, which carries current, satisfies the following equation:

$$\nabla^2 \mathbf{A} = 0. \tag{2}$$

In accordance with the coordinate system assumed (figure 1) the vectorial magnetic potential has only an $A_z$ component which we will call $A$. The magnetic field nulling condition in the superconducting region requires that the normal component of the

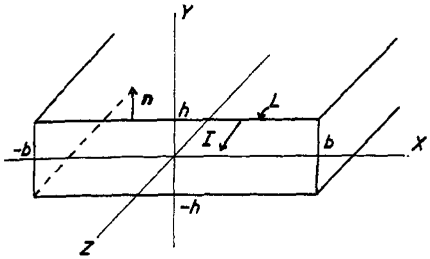


**Figure 1.** Coordinate system assumed for the conductor.

magnetic induction vector $\boldsymbol{B}$ is neutralized on the surface of the superconductor. Taking the dependence $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ into consideration we have

$$\boldsymbol{n} \cdot (\nabla \times \boldsymbol{A}) = \partial A / \partial s. \tag{3}$$

This means the surface of the superconductor may be considered as an equipotential for the vector potential, i.e.

$$A = \text{constant} \,|_{X, Y \in L}. \tag{4}$$

The constant is determined from the normalizing condition

$$\int_L j_\text{p} \, \mathrm{d}s = I. \tag{5}$$

The distribution of the surface current density follows from Ampere's law and from the condition that the magnetic field is neutralized in the superconducting region and is determined by the dependence

$$\boldsymbol{n} \times (\nabla \times \boldsymbol{A}) = \mu_0 j_\text{p} |_{X, Y \in L}. \tag{6}$$

Conditions (4)–(6) give boundary conditions for the component $A$ of the magnetic potential.

In the case under investigation the most effective method of solving equation (2) with conditions (4)–(6) is the conformal representation method in which the external region of the conductor is mapped into the region outside a unit circle; thus the complex magnetic potential is given by

$$\Phi(w) = \frac{\mu_0 I}{2\pi} \ln\left(\frac{1}{w}\right). \tag{7}$$

We introduce dimensionless coordinates

$$x = X/h \tag{8}$$

$$y = Y/h. \tag{9}$$

If we proceed in the same way as Cockcroft (1928) we transform the external region of the rectangle into an external region of a unit circle in two stages. We use the symmetry of the field about the $x$ axis and make a cut along that axis; in the first stage the upper half-plane $v$ is represented on the upper half-plane with a cut-out A′B′C′CBA (figure 2($a$)) using the Christoffel–Schwarz equation

$$\frac{du}{dv}=D\left(\frac{v^2-1}{v^2-1/k}\right)^{1/2}. \tag{10}$$

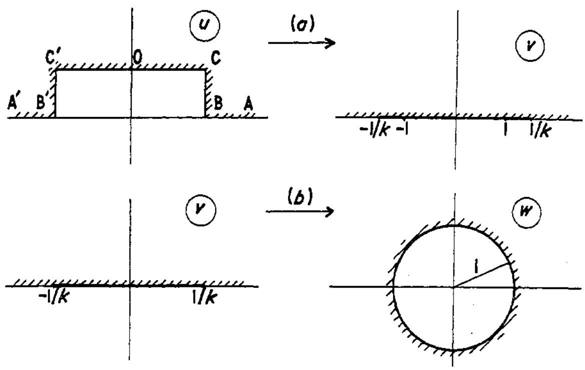


**Figure 2.** Conformal mapping: ($a$) first stage; ($b$) second stage.

Because of the symmetry of the figure A′B′C′CBA about the $y$ axis, the following relations have been assumed between various points on the $u$ and $v$ planes:

| $u$ | $v$ |
|---|---|
| 0 | 0 |
| C | 1 |
| B | $1/k$ |
| A | $\infty$ |

From these relations we can determine the constants $D$ and $k$ in the set of equations:

$$\frac{D}{k}(E(k_1)-k^2F(k_1))=1$$
$$\frac{D}{k}(E(k)-k_1^2F(k))=a \tag{11}$$

where

$$k_1^2=1-k^2 \tag{12}$$
$$a=b/h. \tag{13}$$

In the second stage the external region of the unit circle is represented on the upper half-plane $v$ with a segment cut out, $[-1/k, 1/k]$ (figure 2($b$)), by means of Zukowski's formula:

$$v=\frac{1}{2k}\left(w+\frac{1}{w}\right). \tag{14}$$

Since we know the complex magnetic potential, we can determine the surface current density using

$$j_{\mathrm{P}} = \frac{1}{\mu_0}\left|\frac{\mathrm{d}\Phi}{\mathrm{d}u}\right|_{u\in L}. \tag{15}$$

Using

$$\frac{\mathrm{d}\Phi}{\mathrm{d}u} = \frac{\mathrm{d}\Phi}{\mathrm{d}w}\frac{\mathrm{d}w}{\mathrm{d}v}\frac{\mathrm{d}v}{\mathrm{d}u} \tag{16}$$

and equations (10) and (14) we have

$$j_{\mathrm{P}} = \frac{I}{2\pi h}\left|\frac{1}{D(v^2-1)^{1/2}}\right|. \tag{17}$$

The variables $x$ and $y$ as functions of parameter $v$, which varies within the closed interval $[-1/k, 1/k]$, have been determined from equation (10); namely

$$x = \frac{D}{k}(E(k, \sin^{-1} v) - k_1^2 F(k, \sin^{-1} v)) \qquad \text{for } y = 1 \text{ and } v \in [0, 1] \tag{18}$$

and

$$y = \frac{D}{k}\left(E\left(k_1, \sin^{-1}\frac{[1-(kv)^2]^{1/2}}{k_1}\right) - k^2 F\left(k_1, \sin^{-1}\frac{[1-(kv)^2]^{1/2}}{k_1}\right)\right)$$

$$\text{for } x = a \text{ and } v \in [1, 1/k]. \tag{19}$$

Equations (17)–(19) determine the surface current density in a rectangular conductor made of a superconducting material.

## 3. Current density distribution in a conductor for a rectangular surge

With the aid of quasistationary approximations of Maxwell's equations, an equation for the diffusion of current into the conductor region can be obtained. We start with the equation

$$\nabla^2 j = \mu_0 \sigma \partial j/\partial t. \tag{20}$$

We introduce the dimensionless variables

$$\tau = t/\mu_0 \sigma h^2 \tag{21}$$

$$i = j/J \tag{22}$$

where

$$J = I/4h^2 \tag{23}$$

and use equations (8), (9) and (13) to obtain equation (20) in the dimensionless form

$$\nabla^2 i = \partial i/\partial \tau. \tag{24}$$

The normalizing condition for dimensionless variables has the following form

$$\int_0^a \int_0^1 i \,\mathrm{d}x\,\mathrm{d}y = 1. \tag{25}$$

The initial condition for equation (24), considering equation (17), is

$$i|_{\tau=0_+}=\frac{j_P}{Jh}\begin{cases}\delta(-1+y)+\delta(1+y) & \text{for } x\in[-a,a]\\ \delta(-a+x)+\delta(a+x) & \text{for } y\in[-1,1].\end{cases} \tag{26}$$

The solution of equation (24), with condition (25), gives the current density distribution in a rectangular conductor for a rectangular surge in the following form:

$$i(x,y,\tau)=\frac{1}{a}+\sum_{\substack{m=0\\ m+n\neq 0}}^{\infty}\sum_{n=0}^{\infty}C_{mn}\cos(n\pi y)\cos\left(m\pi\frac{x}{a}\right)\exp\left\{-\left[n^2+\left(\frac{m}{a}\right)^2\right]\pi^2\tau\right\}. \tag{27}$$

A formula for the coefficients in equation (27) is obtained from equations (25), (26) and (17)–(19):

$$C_{mn}=\beta\left\{(-1)^n\int_0^{\sin^{-1}k}\cos\left[m\frac{\pi D}{ak}\left(E\left(k,\sin^{-1}\frac{\sin\alpha}{k}\right)-k_1^2F\left(k,\sin^{-1}\frac{\sin\alpha}{k}\right)\right)\right]\mathrm{d}\alpha\right.$$
$$+(-1)^m\int_{\sin^{-1}k}^{\pi/2}\cos\left[n\frac{\pi D}{k}\left(E\left(k_1,\sin^{-1}\frac{\cos\alpha}{k_1}\right)\right.\right.$$
$$\left.\left.\left.-k^2F\left(k_1,\sin^{-1}\frac{\cos\alpha}{k_1}\right)\right)\right]\mathrm{d}\alpha\right\} \tag{28}$$

where

$$\beta=\begin{cases}8/\pi a & \text{for } m+n\neq 1\\ 4/\pi a & \text{for } m+n=1.\end{cases}$$

In figure 3 we show the distribution of dimensionless current density for time $\tau=0_+$ and in figure 4 we show the distribution of current in time in a rectangular conductor for $a=1$ and $a=10$, calculated by computer.

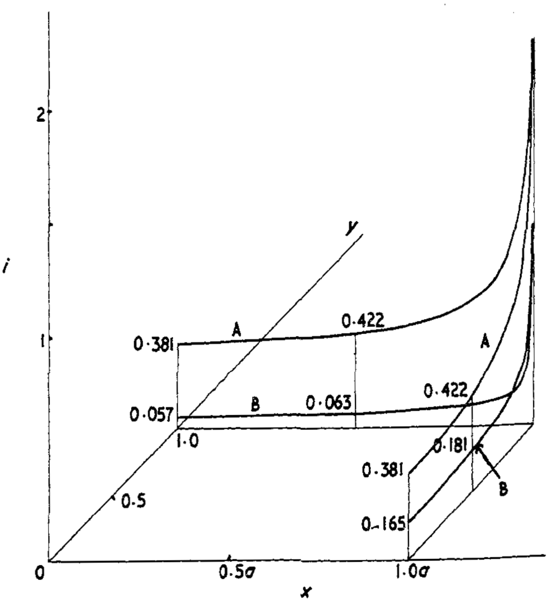


**Figure 3.** Distributions of dimensionless current density for time $t=0_+$. Curves A, $a=1$; curves B, $a=10$.

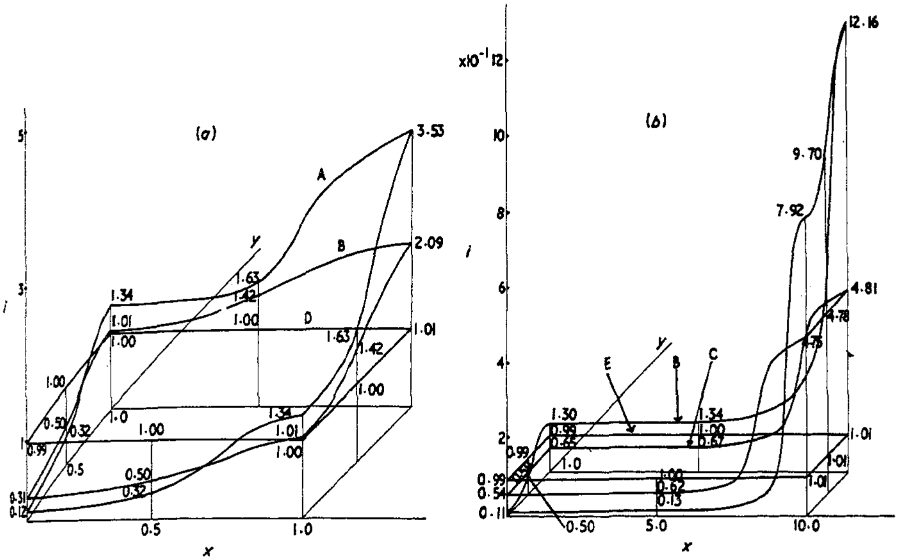


**Figure 4.** Time distributions of dimensionless current density: ($a$) for $a = 1$; ($b$) for $a = 10$. Curve A, $\tau = 0{\cdot}05$; curve B, $\tau = 0{\cdot}1$; curve C, $\tau = 0{\cdot}5$; curve D, $\tau = 1{\cdot}0$; curve E, $\tau = 100{\cdot}0$.

## 4. Current density distribution for a surge of arbitrary shape

The distribution of dimensionless current density for arbitrarily changing current surge $I(t)$ follows from Duhamel's theorem:

$$i(x, y, \tau) = \frac{1}{4h^2 J}\frac{\partial}{\partial \tau}\int_0^\tau i(x, y, \tau - \tau_1) I(\tau_1)\, \mathrm{d}\tau_1. \tag{29}$$

For a linearly increasing surge,

$$I(t) = dt \tag{30}$$

the dimensionless current density distribution is

$$i(x, y, \tau) = \frac{\tau}{a} + \sum_{\substack{m=0\\ m+n\neq 0}}^{\infty} \sum_{n=0}^{\infty} C_{mn} \cos(n\pi y) \cos\left(m\pi \frac{x}{a}\right) \frac{1 - \exp\{-[n^2 + (m/a)^2]\pi^2 \tau\}}{n^2 + (m/a)^2}. \tag{31}$$

In the case of an harmonic current

$$I(\tau) = I_m \sin(\omega_0 \tau), \tag{32}$$

where $\omega_0 = \mu_0 \sigma h^2 \omega$, for $\tau \to \infty$ the distribution of dimensionless current density in a stationary state is

$$i(x, y, \tau) = \frac{\sin \omega_0 \tau}{a} + \sum_{\substack{m=0\\ m+n\neq 0}}^{\infty} \sum_{n=0}^{\infty} C_{mn} \frac{\omega_0 \cos(n\pi y) \cos(m\pi x/a)}{1 + \{\omega_0/[n^2 + (m/a)^2]\pi^2\}}$$
$$\times \left(\frac{\cos(\omega_0 \tau)}{[n^2 + (m/a)^2]\pi^2} + \frac{\omega_0 \sin(\omega_0 \tau)}{\{[n^2 + (m/a)^2]\pi^2\}^2}\right). \tag{33}$$

## 5. Conclusions

From equations (27), (28) and (29) we can determine the distribution of current density in a rectangular conductor for an arbitrary shape of forcing current. The series that appear in the formulae converge slowly and require large numbers of terms, especially for time values less than $\mu_0\sigma h^2/\pi^2$. Thus fast computers are necessary for such calculations.

The diffusion of current into the region of a conductor with cross section different from a square, i.e. $a \neq 1$, proceeds in accordance with the combination of two different time constants, namely $\mu_0\sigma b^2/\pi^2$ and $\mu_0\sigma h^2/\pi^2$. From equation (27), for time values of about $5\mu_0\sigma h^2/\pi^2$, the conductor may be considered in practice to be one dimensional, i.e. the diffusion of current takes place along the side $b$, as in the case of a foil. Thus, in this case, we can use the model proposed by Zimny (1970).

## References


Cockcroft J D 1928 *Proc. R. Soc.* **122** 533–42

Dwight H B 1918 *Trans. AIEE* **37** 1379–400

Forbes H C and Gorman L J 1933 *Elect. Engng* **52** 636–9

Kelvin T W 1890 *Math. Phys. Pap.* **5** 491

Kennelly A E, Laws F A and Pierce P H 1915 *Proc. AIEE* **34** 1953–2021

Landau L and Lifshitz E 1960 *Electrodynamics of Continua* (Warsaw: Państwowe Wydawnictwo Naukowe) (in Polish)

Strutt M J D 1927 *Ann. Phys., Lpz* **83** 979–1000

Zimny P 1970 *Rozpr. Elektrotech.* **16** 481–8 (in Polish)